

Schrödinger cat state of a Bose-Einstein condensate in a double-well potential

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(February 1, 2008)

We consider a weakly interacting coherently coupled Bose-Einstein condensate in a double-well potential. We show by means of stochastic simulations that the system could possibly be driven to an entangled macroscopic superposition state or a Schrödinger cat state by means of a continuous quantum measurement process.

03.75.Fi, 03.65.Bz, 05.30.Jp

I. INTRODUCTION

Since the first observations of Bose-Einstein condensation in dilute alkali-metal atomic gases [1–3] the ultra-cold atomic gases have stimulated significant theoretical and experimental interest [4]. The scientific progress has been rapid and examples of recent experiments include the development of accurate detection methods [5], the state preparation of topological structures [6], and the applications in nonlinear atom optics [7]. Due to the macroscopic quantum coherence Bose-Einstein condensates (BECs) could possibly be also used in the future as a test for the foundations of quantum mechanics. One particularly puzzling and controversial issue has been the existence of macroscopic quantum superposition states in many-particle quantum systems. In this paper we propose a method of creating the Schrödinger cat states of different atom occupation numbers in a weakly interacting BEC confined in a double-well potential.

The existence of the superpositions of macroscopically distinguishable states in BECs has been addressed by several authors [8–12]. The superposition state may arise as the ground state of a coherently coupled BEC in a double-well potential [9,10]. Under certain conditions it could be reached as a result of a unitary time evolution [11]. Previously, we proposed a method of creating Schrödinger cat states in BECs by means of scattering light from two BECs moving with opposite velocities [8]. The nonunitary evolution due to the detections of scattered photons drives the condensates to macroscopic quantum superposition states. In this paper we show that a continuous quantum measurement process could also drive a trapped coherently coupled BEC in a double-well potential to a Schrödinger cat state. The advantage of the proposed scheme is that the BEC is almost stationary and trapped. Moreover, as a result of the back-action of quantum measurement process the superposition state could be reached rapidly unlike in a slow unitary evolution, which may be very sensitive to decoherence.

The paper is organized as follows: We begin in Sec. II A by introducing the unitary system Hamiltonian. The scattering of light and the measurement geometry is described in Sec. II B. In Sec. II C we study the dynamics of the open quantum system in terms of stochastic trajectories of state vectors. The results of the numerical simulations are presented in Sec. III. Finally, a few concluding remarks are made in Sec. IV.

II. SYSTEM DYNAMICS

A. Unitary evolution

We consider the evolution of a BEC in a double-well potential in a two-mode approximation. Macroscopic quantum coherence of BECs results in coherent quantum tunneling of atoms between the two modes representing ‘two BECs’. This is analogous to the coherent tunneling of Cooper pairs in a Josephson junction [13–18]. To obtain the system Hamiltonian in the two-mode approximation for the unitary evolution of the BEC we approximate the total field operator by the two lowest quantum modes $\psi(\mathbf{r}) \simeq \psi_b(\mathbf{r})b + \psi_c(\mathbf{r})c$, where ψ_b and ψ_c stand for the local mode solutions of the individual wells with small spatial overlap. The corresponding annihilation operators are denoted by b and c . The Hamiltonian in the two-mode approximation reads [14]:

$$\frac{H_S}{\hbar} = \xi b^\dagger b + \Omega(b^\dagger c + c^\dagger b) + \kappa[(b^\dagger)^2 b^2 + (c^\dagger)^2 c^2]. \quad (1)$$

Here ξ is the energy difference between the modes. The tunneling between the two wells is described by Ω , which is proportional to the overlap of the spatial mode function of the opposite wells. The short-ranged two-body interaction strength is obtained from $\kappa = 2\pi a\hbar/m \int |\psi_b(\mathbf{r})|^4$, where a and m denote the scattering length and the atomic mass, respectively. For simplicity, here we have assumed that $\int |\psi_b(\mathbf{r})|^4 = \int |\psi_c(\mathbf{r})|^4$. A necessary condition for the validity of the single-mode approximation in a harmonic trap is that the oscillation energy of the atoms does not dominate over the mode energy spacing of the trap.

According to the Josephson effect, the atom numbers of the BECs determined by the Hamiltonian (1) may oscillate even if the number of atoms in each well is initially equal. Due to the nonlinear self-interaction the number oscillations also exhibit collapses and revivals. These have been studied numerically in Ref. [14]. We may also

obtain a simple analytical description by solving the dynamics in the rotating wave approximation in the limit $\Omega \gg N\kappa$ as described in Ref. [19]. Here N denotes the total number of atoms. In particular, we may solve the number of atoms $N_b \equiv \langle b^\dagger b \rangle$ in well b . We consider a coherent state in the both wells as an initial state. Then we obtain

$$N_b = \frac{N}{2} [1 + e^{N[\cos(\kappa t) - 1]} (\sqrt{1 - \beta^2} \cos \eta - \beta \sin \varphi \sin \eta)], \quad (2)$$

with $\eta \equiv N\beta \sin(\kappa t) \cos \varphi - 2\Omega t$. Here all the operators on the right-hand side have been evaluated at $t = 0$. It is useful to define the real expectation values β and φ in the following way:

$$\beta e^{i\varphi} \equiv \frac{2}{N} \langle b^\dagger c \rangle. \quad (3)$$

For a coherent state with equal atom numbers in the two wells we obtain the visibility $\beta = 1$. The relative phase between the wells is φ . For a number state there is no phase information and $\beta = 0$. For unequal atom numbers the maximum visibility is $\beta_{\max} = 2(N_b N_c)^{1/2}/N$. We see that the number of atoms in Eq. (2) may oscillate in the case of initially equal atom numbers $\beta = 1$. The amplitude of sinusoidal oscillations, representing the macroscopic coherence, collapses. For instance, for $\varphi = \pi/2$ and $\beta = 1$ we may obtain the short time decay by considering the time scales $N\kappa t \ll 1 \ll \Omega t$. Then the decay of the oscillations has the form $\exp(-N\kappa^2 t^2/2)$. This is the rate of the phase diffusion and it may be interpreted as the width of the relative phase $\langle \Delta\varphi^2(t) \rangle \simeq N\kappa^2 t^2 \sim \kappa^2 t^2 / \langle \Delta\varphi^2(0) \rangle$. Perhaps surprisingly the functional dependence of the width in this case is the same as in the case of two uncoupled BECs [4].

If the phase is unknown we may obtain the ensemble average by integrating over the relative phase φ in Eq. (2):

$$N_b = \frac{N}{2} \{1 + e^{N[\cos(\kappa t) - 1]} \sqrt{1 - \beta^2} \times \cos(2\Omega t) J_0[N\beta \sin(\kappa t)]\}, \quad (4)$$

where J_0 is the 0th order Bessel function. If the both wells have initially equal number of atoms, $\beta = 1$, the atom numbers do not oscillate. For unequal atom numbers the oscillations collapse at the first zero of the Bessel function $t \simeq 2.4/(N\beta\kappa)$ for $(N_b N_c)^{1/2} \gg 1$.

B. Quantum measurement process

The time evolution of the system is nonunitary, when we include the effect of quantum measurement process. We consider the nondestructive measurement of the number of atoms in the both wells by means of shining coherent light beams through the atom clouds. The scattered

light beams are combined by a 50-50 beam splitter. We display the measurement setup in Fig. 1.

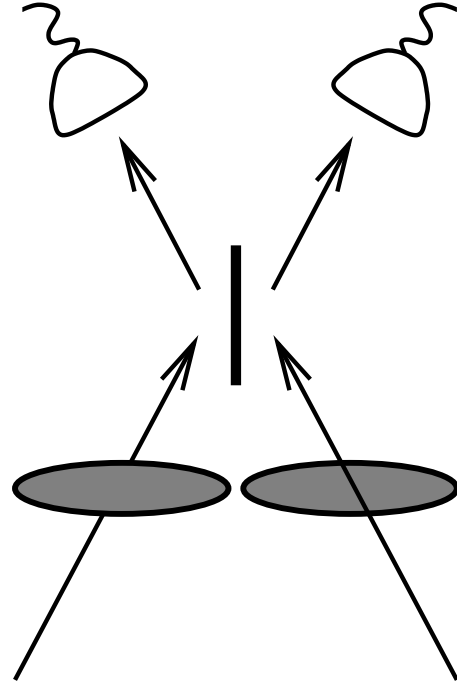


FIG. 1. The measurement setup. Two incoming light fields are scattered from two coherently coupled BECs. The two scattered photon beams are combined by a 50-50 beam splitter. The photons are detected from the two output channels of the beam splitter. One of the output channels introduces only a constant phase shift and it may be ignored.

We assume that the incoming light fields are detuned far from the atomic resonance. For instance, if the shape of the gas is flat and the light is shone through a thin dimension, the multiple scattering is negligible and the sample can be considered optically thin. A BEC atom scatters back to the BEC via coherent spontaneous scattering, stimulated by a large number of atoms in the BEC. Coherently scattered photons are emitted into a narrow cone in the forward direction. By spontaneous scattering we mean that the emission is not stimulated by light, although it is stimulated by atoms. The decay into noncondensate center-of-mass states is also stimulated by the Bose-Einstein statistics. However, at very low temperatures this stimulation is much weaker because most of the particles are in the BEC. As a first approximation we ignore the scattering from and to the noncondensate modes. Then the measurement is nondestructive in the sense that BEC atoms in the modes b and c scatter back to the same modes b and c . Because the overlap of the mode functions of the different wells is assumed to be small, the scattering between the two wells is ignored.

The detection rate of photons on the detectors is the intensity of the scattered light $I(\mathbf{r})$ integrated over the scattering directions divided by the energy of a photon $\hbar ck$. Here k and c stand for the wave number and the

velocity of light. We obtain the detection rate at the channel j :

$$\gamma_j = \frac{1}{\hbar c k} \int d\Omega_{\hat{\mathbf{n}}} r^2 I_j(\mathbf{r}) = 2\Gamma \langle C_j^\dagger C_j \rangle. \quad (5)$$

The photon annihilation operator at the output channel j of the beam splitter is denoted by C_j . For a symmetric measurement geometry we obtain

$$C_1 = \frac{1}{\sqrt{2}}(b^\dagger b - c^\dagger c), \quad C_2 = \frac{1}{\sqrt{2}}(b^\dagger b + c^\dagger c). \quad (6)$$

Because the total number of atoms is assumed to be conserved, the operator $\hat{N} \equiv b^\dagger b + c^\dagger c$ contributes to the measurements only through a constant phase shift. Therefore, we may ignore the effect of the scattering channel 2 on the dynamics.

The scattered intensity may be written in terms of the positive frequency component of the scattered electric field $\mathbf{E}^+(\mathbf{r})$

$$I(\mathbf{r}) = 2c\epsilon_0 \langle \mathbf{E}^-(\mathbf{r}) \cdot \mathbf{E}^+(\mathbf{r}) \rangle, \quad (7)$$

Here ϵ_0 denotes the permittivity of the vacuum.

We assume that the driving electric fields may be approximated by plane waves $\mathbf{E}_d^+(\mathbf{r}) = \mathcal{E} \hat{\mathbf{e}} e^{i(\mathbf{k} \cdot \mathbf{r} - \Omega t)}/2$. In the limit of large atom-light detuning Δ we use the first Born approximation and write the electric fields in the far radiation zone ($kr \gg 1$). Then the scattered field from the well b has the following form [20]:

$$\begin{aligned} \mathbf{E}^+(\hat{\mathbf{n}}r) &= \frac{k^2 \mathcal{R} e^{ikr}}{4\pi\epsilon_0 \Delta r} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{d}) \\ &\int d^3r' e^{i(\mathbf{k} - k\hat{\mathbf{n}}) \cdot \mathbf{r}'} |\psi_b(\mathbf{r}')|^2 b^\dagger b. \end{aligned} \quad (8)$$

Here we have defined the Rabi frequency \mathcal{R} of the atomic dipole matrix element \mathbf{d} by $\mathcal{R} \equiv d\mathcal{E}/(2\hbar)$. We also assumed that $\mathbf{d} \cdot \hat{\mathbf{e}} = d$. In the limit that the characteristic length scale ℓ of the BECs is much larger than the inverse of the wave number of the incoming light $\ell \gg 1/k$, the momentum of the scattered photon is approximately conserved, and we obtain in Eq. (8):

$$\int d^3r' e^{i(\mathbf{k} - k\hat{\mathbf{n}}) \cdot \mathbf{r}'} |\psi_b(\mathbf{r}')|^2 \simeq \delta(k\hat{\mathbf{n}} - \mathbf{k})$$

In this simple case the scattering rate Γ may be easily evaluated:

$$\Gamma = \frac{3\gamma \mathcal{R}^2}{8\pi \Delta^2}, \quad (9)$$

Here $\gamma = d^2 k^3 / (6\pi \hbar \epsilon_0)$ denotes the optical linewidth of the atom.

C. Stochastic Schrödinger equation

The dissipation of energy from the quantum system of macroscopic light fields and the BEC in a double-well potential is described by the coupling to a zero temperature reservoir of vacuum modes, resulting in a spontaneous emission linewidth for the atoms. The dynamics of the continuous quantum measurement process may be unraveled into stochastic trajectories of state vectors [21–23]. The procedure consists of the evolution of the system with a non-Hermitian Hamiltonian H_{eff} , and randomly decided quantum ‘jumps’. In our case the quantum jumps correspond to the detections of spontaneously emitted photons. The system evolution is thus conditioned on the outcome of a measurement. The non-Hermitian Hamiltonian has the following form:

$$H_{\text{eff}} = H_S - i\hbar \Gamma C_1^\dagger C_1, \quad (10)$$

where the unitary system Hamiltonian H_S is determined by Eq. (1).

Equation (10) corresponds to the modification of the state of the system associated with a zero detection result for scattered photons. Because the output is being continuously monitored, we gain information about the system even if no photons have been detected.

The Hamiltonian H_{eff} determines the evolution of the state vector $\psi_{\text{sys}}(t)$. If the wave function $\psi_{\text{sys}}(t)$ is normalized, the probability that a photon from the output channel 1 of the beamsplitter is detected during the time interval $[t, t + \delta t]$ is

$$P(t) = 2\Gamma \langle \psi_{\text{sys}}(t) | C_1^\dagger C_1 | \psi_{\text{sys}}(t) \rangle \delta t. \quad (11)$$

We implement the simulation algorithm as follows: At the time t_0 we generate a quasi-random number ϵ which is uniformly distributed between 0 and 1. We assume that the state vector $\psi_{\text{sys}}(t_0)$ at the time t_0 is normalized. Then we evolve the state vector by the non-Hermitian Hamiltonian H_{eff} iteratively for finite time steps Δt . At each time step n we compare ϵ to the reduced norm of the wave function, until

$$\langle \psi_{\text{sys}}(t_0 + n\Delta t) | \psi_{\text{sys}}(t_0 + n\Delta t) \rangle < \epsilon,$$

when the detection of a photon occurs. If the photon has been observed during the time step $t \rightarrow t + \Delta t$ we take the new wave function at $t + \Delta t$ to be

$$| \psi_{\text{sys}}(t + \Delta t) \rangle = \sqrt{2\Gamma} C_1 | \psi_{\text{sys}}(t) \rangle, \quad (12)$$

which is then normalized.

III. NUMERICAL RESULTS

We simulate the effect of the system Hamiltonian H_S and the quantum measurement process of scattered photons by means of the stochastic Schrödinger equation.

For simplicity, we set the total number of atoms to be reasonably small $N = 200$. We start from a slightly asymmetric initial state with the two modes in number states $N_b = 102$ and $N_c = 98$. We choose $\Gamma/\Omega = 5 \times 10^{-6}$.

After just a few detected photons we observe the emergence of two well-separated amplitude maxima in the occupation number of atoms in one of the two wells. These correspond to a macroscopic number state superposition or a Schrödinger cat state. Because the total number of atoms is assumed to be conserved, the atom numbers in the two wells are entangled and we have a Bell-type of superposition state. In Fig. 2 we display the absolute value of the wave function $|\psi_b|$ in mode b in number state basis in a single run at two different times. In this case the nonlinearity vanishes $\kappa = 0$ and $\xi/\Omega = 0.1$. We clearly recognize the two distinct peaks in the number distribution. For instance, the peaks in Fig. 2 (b) are centered at $N_b \simeq 10$ and $N_b \simeq 190$ corresponding to maxima at $N_c \simeq 190$ and $N_c \simeq 10$, respectively. In Fig. 3 we show the absolute value of the wave function for a different run with $N\kappa/\Omega = 0.2$ and $\xi/\Omega = 0.001$.

We also describe the state of the BEC in terms of the quasiprobability Q distribution. For the number state distribution of atoms $|\psi_b\rangle = \sum_n c_n |n\rangle$ in mode b we obtain [24]:

$$Q(\alpha) = \frac{|\langle \alpha | \psi_b \rangle|^2}{\pi} = \frac{e^{-|\alpha|^2}}{\pi} \left| \sum_{n=0}^N \frac{\alpha^n c_n^*}{\sqrt{n!}} \right|^2. \quad (13)$$

The Q function represents the phase-space distribution. The amplitude and phase quadratures are denoted by X and Y . In polar coordinates the radius in the xy plane is equal to $N_b^{1/2}$ and the polar angle is the relative phase between the atoms in the two wells. In Fig. 4 we show the Q function distribution of the number state superposition displayed in Fig. 2(a).

It is interesting to emphasize that the measurement of the number of atoms in only one of the wells affects the system dynamics quite differently. In that case the number state distribution remains well localized and approximately approaches a coherent state [17]. Even though we start from a number state with no phase information, the detections of spontaneously scattered photons establish a macroscopic coherence or the off-diagonal long-range order (ODLRO) between the atoms in the two separate wells. This is similar to establishing the coherence between two BECs as a result of the counting of atoms [25,26]. However, in the present case the continuous measurement process drives the system to a Schrödinger cat state and the ODLRO remains small. We may describe the visibility of the macroscopic coherence between the two wells by the real parameter β defined in Eq. (2). We show the relative visibility $\beta_r \equiv \beta/\beta_{\max}$ and the number of atoms in well b as a function of the number of measurements for $\kappa = 0$ in Fig. 5 and for $N\kappa/\Omega = 0.2$ in Fig. 6. Due to the emergence of the superposition state the visibility remains below one. The measurement process of

the scattered photons significantly complicates the dynamics of the number of atoms predicted by the unitary time evolution of Eq. (1).

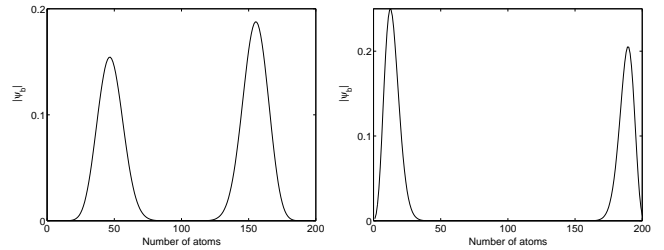


FIG. 2. The entangled Schrödinger cat states of atoms with different atom numbers. We show the absolute value of the wave function $|\psi_b|$ in the number state basis for the atoms in well b during one realization of stochastic measurement process of spontaneously scattered photons after (a) 750 and after (b) 1700 detections. The two maxima correspond to the superposition states. The nonlinearity $\kappa = 0$ and the total number of atoms $N = 200$.

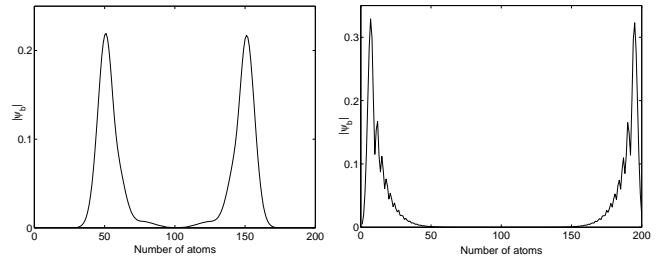


FIG. 3. The entangled Schrödinger cat states of atoms with different atom numbers. We show the absolute value of the wave function $|\psi_b|$ during one realization of stochastic trajectory after (a) 100 and after (b) 1250 detections. The nonlinearity $N\kappa = 0.2$.

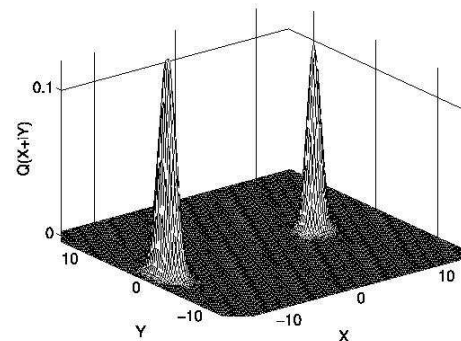


FIG. 4. The Q function of the Schrödinger cat state. We show the Q function of the quantum state displayed in Fig. 2 (a). The two peaks are located at different value of the radius $N_b^{1/2}$ representing the different maxima of the occupation numbers.

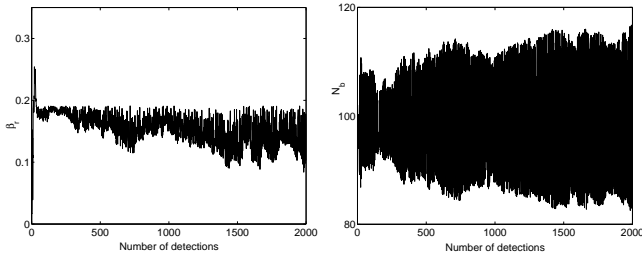


FIG. 5. The (a) relative visibility of the interference β_r and the (b) number of atoms N_b in one of the wells as a function of the number of detected photons. Due to the Schrödinger cat state the visibility never reaches one. Here the nonlinearity $\kappa = 0$.

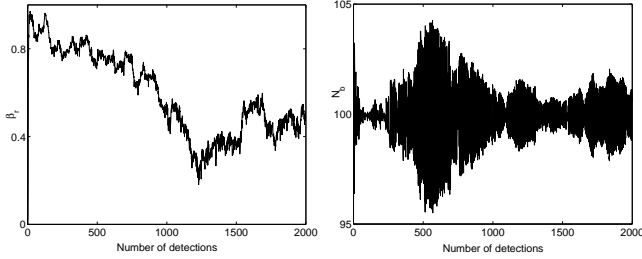


FIG. 6. The (a) relative visibility of the interference β_r and the (b) number of atoms N_b in one of the wells as a function of the number of detected photons. The nonlinearity $N\kappa/\Omega = 0.2$.

IV. FINAL REMARKS

We studied the generation of the macroscopic superposition states or the Schrödinger cat states of a BEC in a double-well potential. The Schrödinger cat state was shown to emerge as a result of the continuous quantum measurement process of scattered photons. The particular detection geometry increases the fluctuations of the relative atom number between the two wells. Therefore the superposition states are more stable in the detection process. The proposed setup is an open quantum system and the creation of the Schrödinger cat state in this case is not based on reaching the ground state of a BEC in a double-well potential [9,10]. The advantage over previously proposed open systems schemes [8] is that the BEC is stably trapped and the superposition state for a small BEC could be created by scattering only a few photons.

In the present discussion we ignored the effect of decoherence [27]. The interaction of the BEC with its environment results in the decoherence of the superposition states. We can identify several sources of decoherence. Decoherence by amplitude damping or by phase damping has been estimated in Ref. [28]. The inelastic two-body and three-body collisions between the condensate atoms and the noncondensate atoms change the number of condensate atoms and introduce amplitude damping. The

phase damping corresponds, e.g., to elastic collisions between the condensate and noncondensate atoms in which case the number of BEC atoms is conserved. If the number of atoms in a BEC is not large, the scattering between the condensate and noncondensate atom fractions may not be negligible. This also introduces amplitude decoherence. Additional sources of decoherence may be, e.g., the imperfect detection of the scattered photons and the fluctuations of the magnetic trap. In Ref. [12] it was proposed that the decoherence rate of a BEC could be dramatically reduced by symmetrization of the environment and by changing the geometry of the trapping potential to reduce the size of the thermal cloud. Moreover, the continuous measurement process increases the information about the system and therefore it could also reduce the decoherence rate.

ACKNOWLEDGEMENTS

We are indebted to the late Prof. Walls for his support, inspiration, and encouragement during his last years. This work was financially supported by the EC through the TMR Network ERBFMRXCT96-0066.

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